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Low Lying Spectrum of Weak-Disorder Quantum Waveguides

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Abstract We study the low-lying spectrum of the Dirichlet Laplace operator on a randomly wiggled strip. More precisely, our results are formulated in terms of the eigenvalues of finite segment approximations of the infinite waveguide. Under appropriate weak-disorder assumptions we obtain deterministic and probabilistic bounds on the position of the lowest eigenvalue. A Combes-Thomas argument allows us to obtain a so-called 'initial length scale decay estimate' as they are employed in the proof of spectral localization using the multiscale analysis method.

Keywords Random Hamiltonian · Weak disorder · Random geometry · Quantum waveguide · Low-lying spectrum · Asymptotic analysis · Anderson localization

1 Introduction

The propagation of waves in disordered media can be modeled using differential equations governed by a random Hamiltonian. The most important questions in this context concern the long time behavior of waves which oftentimes allows to conclude results concerning the transport properties of the material described by the random operator. A particularly well studied class of operators is the one which arises in the quantum mechanical description of disordered solids. To this class belong various types of random Schrödinger operators, e.g.

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the Anderson model on $\ell^2(\mathbb{Z}^d)$, or a Laplacian with Poisson distributed repulsive impurity potentials on $L^2(\mathbb{R}^d)$.

For most operators it is possible to relate propagation properties to spectral features, e.g. by the use of the RAGE or Ruelle theorem. From this point of view it is justified to study first the measure theoretic spectral types which arise in a certain random model, and then relate them to the transport properties of the considered material. This leads to the study of a plethora of spectral features, some of which are specific to the situation that we are dealing with a random operator, i.e. with an infinite family of individual operators. Let us mention some of these properties: the characteristics of the spectrum as a subset of the real line, its band structure, location of the spectral infimum (i.e. the infimum of the bottom of the spectrum when varied over all members of the family), the density of the spectrum in various energy regions, decay properties of the Green's function, etc.

To be able to point out the interesting contribution of the paper at hand, we shall assume in the further discussion that the randomness enters the Hamiltonian via a countable family of random variables. Already when considering the very basic features of the spectrum, one sees that it makes a great difference whether the operator (more precisely: the associated quadratic form) depends in a monotone or non-monotone way on the random variables entering the model. In the case of monotone dependence the spectral minimum of the operator family obviously corresponds to the configuration where all random variables are set to one of the extremal values. Similarly, in the monotone situation the band structure of the spectrum can be analyzed using rather basic sandwiching arguments, see e.g. [17]. It is consistent with these elementary examples of the advantages of monotonicity that there is a rather good understanding of typical energy/disorder regimes where monotone models exhibit localization of waves, see the monographs and survey articles [15, 16, 31, 39].

If the dependence of the quadratic form on the random variables is not monotone, already the identification of the spectral minimum is not obvious and sometimes a highly non-trivial question, see e.g. [1, 23]. For more intricate properties, like the regularity of the density of states or the analysis of spectral fluctuation boundaries, the difference between monotone and non-monotone models is even more striking.

Nevertheless there has been a continuous effort to bring the understanding of models with non-monotone dependence on the randomness to a similar level as the one for monotone models. The model of this type to which most attention was devoted so far is the alloy type Hamiltonian with single site potentials of changing sign, see e.g. [12, 20, 21, 23, 25, 32, 38]. More recently also the discrete analog of this model was studied in [7, 26, 33, 40, 41]. Electromagnetic Schrödinger operators with random magnetic field [4, 12, 24, 34, 35, 37], as well as Laplace-Beltrami operators with random metrics [27–29] are other examples with non-monotonous parameter dependence.

A very interesting model with geometric disorder is the random displacement model, which exhibits also no obvious monotonicity, cf. e.g. [1, 19, 22]. Another relevant model (although not defined in terms of a countable family of i.i.d. random variables) without obvious monotonicity is a random potential given by a Gaussian stochastic field with sign-changing covariance function, c.f. [13, 36, 42].

In this paper we consider a family of Hamiltonians which consists of the Dirichlet Laplacian on a randomly wiggled waveguide. In this model the dependence of the quadratic form on the random variables is neither monotone nor linear. In this respect it is related to the random displacement model. Moreover, in our model, the randomness does not enter via potential terms, but rather through differential operator terms. A random waveguide model has been studied before in [18]. There the randomness enters via a variation of the width of the waveguide. This type of perturbation leads to a quadratic form which depends monotonously on the random variables and is thus structurally different from our model.

There is a substantial body of literature devoted to the analysis of eigenvalues below the essential spectrum of bent, asymptotically straight waveguides, see for instance [8]. These eigenvalues are in contrast to the purely absolutely continuous spectrum exhibited by an straight waveguide. Thus a local geometric perturbation leads to the emergence of discrete eigenvalues. Given this fact, it is interesting to ask whether geometric perturbations which are ergodic and random lead to dense point spectrum below the continuous one, in analogy to the phenomenon encountered for several classes of random Schrödinger operators mentioned above. One should point out that in the present paper the local geometric perturbations of the waveguide are introduced in a somewhat different way than in [8].

Let us now describe the main result of this paper. We derive lower bounds on the first eigenvalue for a finite segment of a randomly wiggled strip in $\mathbb{R}^2$. They measure how far the eigenvalue may move up, if the vector of random variables moves away from the optimal configuration. As an application we obtain a second result: In the terminology of the *multi-scale analysis* (MSA) it is a *initial length scale estimate* for energies near the bottom of the spectrum in the weak disorder regime. It corresponds to the induction anchor of the MSA. This should be understood as a step towards a localization proof using the MSA. If there would be an appropriate Wegner estimate at disposal at low energies (which we don't have at the moment) an adaptation of the usual MSA, e.g. as presented in [10, 11, 31] would lead to localization.

Let us say a few words about our methods of proof. It consists of a deterministic and probabilistic part. For a finite segment of the waveguide one can use methods from asymptotic analysis to estimate the position of the principal eigenvalue of the Laplacian. In this situation only a finite number of random variables enters the operator. It is this part which requires the weak disorder restriction. The mentioned results can be combined with a Dirichlet-Neumann bracketing argument and a large deviations principle to arrive at an exponential probabilistic bound on the position of the lowest finite segment eigenvalue. Using a Combes-Thomas estimate [2, 5, 6, 31] this can be turned into an off-diagonal decay estimate on the Green's function, which plays the role of the initial length scale estimate in the MSA.

It is maybe worthwhile to point out some differences to the recent paper [23] of Klopp and Nakamura which is devoted to the proof of Lifshitz tails for alloy-type Schrödinger operators with single site potentials which are allowed to change sign. There are two aspect in common between this work and ours: both of them concern the analysis of the low lying eigenvalues of finite volume random Hamiltonians and both of them deal with non-monotone parameter dependence. There are also two main differences: we are not able to give an Lifshitz bound on the integrated density of states for our model, since the global disorder coupling constant has to be chosen dependent on the volume scale. If one lets the scale tend to infinity the global coupling constant has to go to zero. On the other hand we assume no reflection symmetry for the individual perturbations. A crucial assumption of [23] is that the single site potential obeys such an condition. It allows Klopp and Nakamura to use, after some work and ingenious ideas, an effective decoupling between different random variables (similarly as in the case of fixed-sign single site potentials). This means that it is only necessary to perform perturbation theory with respect to one coupling constant. This aspect of the proof of [23] is discussed on p. 1134 before the statement of hypothesis (H2) there. In our model there is no such symmetry assumption. This means that the analysis of the single parameter random Hamiltonian on a unit cell with Neumann b.c. does not give us the crucial information which was instrumental in the proof strategy of [23]. Consequently, we need to analyze the fully interacting model, which leads to an eigenvalue perturbation problem with respect to many parameters.

The paper is organized as follows: in the next section we define rigorously our model, in Sect. 3 we state the probabilistic estimates on the position of the principal eigenvalue of a finite segment of a random waveguide and on the exponential off-diagonal decay of the associated Green's function, in Sect. 4 we reduce the proof of the two above statements to deterministic bounds on the first eigenvalue, and in the final Sect. 5 the mentioned deterministic estimates are established.

2 Model

We consider random quantum waveguides in $\mathbb{R}^2$ determined by the following data: Let $(\omega_k)_{k\in\mathbb{Z}}$ be a sequence of independent, identically distributed, non-negative, bounded, non-trivial random variables, $\kappa > 0$ a global coupling constant, $l \geq 1$ the length of one (periodicity) cell of the waveguide, and $g \in C_0^2(0,l)$ a single bump function. The following function $G\colon \mathbb{R}\times\Omega \to \mathbb{R}$ determines the shape of the waveguide

$$G(x_1,\omega) := \sum_{k\in\mathbb{Z}} \omega_k\, g(x_1 - kl).$$

Note that $\kappa G(x_1,\omega) = G(x_1,\kappa\omega)$. Together with the global coupling constant $\kappa > 0$ it defines an infinite waveguide as the set

$$\begin{aligned} D_{\kappa,\omega} :&= \{x \in \mathbb{R}^2 \mid x_1 \in \mathbb{R}, \kappa G(x_1,\omega) < x_2 < \kappa G(x_1,\omega) + \pi\} \\ &= \{x \in \mathbb{R}^2 \mid x_1 \in \mathbb{R}, G(x_1,\kappa\omega) < x_2 < G(x_1,\kappa\omega) + \pi\} = D_{1,\kappa\omega} \end{aligned}$$

Our results are formulated in terms of spectral features of finite segments of the infinite waveguide. We define them next. For $N \in \mathbb{N}$ and $j \in \mathbb{Z}$ set

$$D_{\kappa,\omega}(N,j) := \{x \in \mathbb{R}^2 \mid jl < x_1 < (j+N)l, \kappa G(x_1,\omega) < x_2 < \kappa G(x_1,\omega) + \pi\}.$$

Denote by $\Gamma_{\kappa,\omega}(N,j)$ the upper and lower part of the boundary of $D_{\kappa,\omega}(N,j)$, i.e.,

$$\begin{aligned} \Gamma_{\kappa,\omega}(N,j) &:= \{x \in \mathbb{R}^2 \mid jl < x_1 < (j+N)l, x_2 = \kappa G(x_1,\omega)\} \\ &\quad \cup \{x \in \mathbb{R}^2 \mid jl < x_1 < (j+N)l, x_2 = \kappa G(x_1,\omega) + \pi\}. \end{aligned}$$

The remaining part of the boundary $\partial D_{\kappa,\omega}(N,j) \setminus \Gamma_{\kappa,\omega}(N,j)$ is denoted by $\gamma_{\kappa,\omega}(N,j)$.

Let $\mathcal{H}_{\kappa,\omega}(N,j)$ denote the negative Laplace operator on $D_{\kappa,\omega}(N,j)$ with Dirichlet boundary conditions on $\Gamma_{\kappa,\omega}(N,j)$ and Neumann b.c. on $\gamma_{\kappa,\omega}(N,j)$. The lowest eigenvalue of $\mathcal{H}_{\kappa,\omega}(N,j)$ is denoted by $\lambda_{\kappa,\omega}(N,j)$. If $j=0$ we shall use the following shorthand notation:

$$\begin{aligned} &D_{\kappa,\omega}(N) := D_{\kappa,\omega}(N,0), \qquad \Gamma_{\kappa,\omega}(N) := \Gamma_{\kappa,\omega}(N,0), \qquad \gamma_{\kappa,\omega}(N) := \gamma_{\kappa,\omega}(N,0), \\ &\mathcal{H}_{\kappa,\omega}(N) := \mathcal{H}_{\kappa,\omega}(N,0), \qquad \lambda_{\kappa,\omega}(N) := \lambda_{\kappa,\omega}(N,0). \end{aligned}$$

Similarly as for the infinite waveguide we have $D_{\kappa,\omega}(N,j) = D_{1,\kappa\omega}(N,j)$. Since $\kappa > 0$ is arbitrary we may assume without restricting the model

$$\max\{\|g\|_{C[0,l]}, \| \|g'\|_{C[0,l]}, \|g''\|_{C[0,l]}\} = 1. \tag{2.1}$$

Denote the distribution measure of ω_k by μ. It will be convenient to think of μ as a measure on the semiaxis $[0, \infty)$ with support in the unit interval $[0, 1]$. Thus any ω_k takes values larger than 1 only with zero probability. Then $\mathbb{P} = \bigotimes_{k\in\mathbb{Z}} \mu$ denotes the product measure on the configuration space $\Omega = \times_{k\in\mathbb{Z}}[0, \infty)$ whose elements we denote by $\omega = (\omega_k)_{k\in\mathbb{Z}}$.

Note that by the assumptions on μ the following statements hold for $\mathbb{P}$-almost all $\omega \in \Omega$: $\omega \in \ell^\infty(\mathbb{Z})$ and $\|\omega\|_\infty := \sup_{k\in\mathbb{Z}} |\omega_k| = \sup_{k\in\mathbb{Z}} \omega_k < \infty$. On appropriate subspaces of Ω we define the usual ℓ^p-norms, in particular $\|\omega\|_1 := \sum_{k\in\mathbb{Z}} \omega_k$ and $\|\omega\|_2 := (\sum_{k\in\mathbb{Z}} \omega_k^2)^{\frac{1}{2}}$.

The randomness of the finite segments $D_{\kappa,\omega}(N, j)$ arises from only a finite number of random variables (ω_k). For this reason it is convenient to have the following notation at disposal: For $\Lambda \subset \mathbb{Z}$ define the projection map

$$\pi_\Lambda : \Omega \to \Omega, \quad (\pi_\Lambda(\omega))_k = \omega_k\, \chi_\Lambda(k)$$

Here χ_A denotes the indicator function of a set A. If $\Lambda \subset \mathbb{Z}$ is finite $\pi_\Lambda(\Omega)$ is contained $\ell^p(\mathbb{Z})$ for every $p \in [1, \infty]$. We shall use the shorthand notation $\|\omega\|_{\Lambda,p} := \|\pi_\Lambda \omega\|_p$ and in the case $\Lambda = \{1, \dots, N\}$

$$\|\omega\|_{N,p} := \|\omega\|_{\Lambda,p} := \|\pi_\Lambda \omega\|_p.$$

3 Probability of Low Lying Eigenvalues and the Initial Length Scale Estimate

Here we present estimates on the probability that the lowest eigenvalue of $\mathcal{H}_{\kappa,\omega}(N, j)$ is close to one. The first information which one has is that the minimum of the spectrum of the Laplacian on an straight waveguide segment $\mathcal{H}_{0,\omega}(N, j)$ is equal to one. This follows directly from separation of variables. We shall see later that no operator $\mathcal{H}_{\kappa,\omega}(N, j)$ has spectrum below one. In this sense we can say that one is the minimal spectral value for all random configurations.

To formulate the main result we shall use the following quantities:

$$\tilde{g} := g - \frac{1}{l}\int_0^l g(t)\,\mathrm{d}t, \qquad c_2 = \frac{3\,\|\tilde{g}\|_{L_2(0,l)}}{2\,l^3}, \qquad c_3 = \frac{3\,\|\tilde{g}\|^2_{L_2(0,l)}}{5000\,l^7}.$$

Theorem 3.1 *Let g and μ as above be given. Let $\gamma > 34$. Then there exists an initial scale N_1 such that if $N \geq N_1$ the interval*

$$I_N := \left[\frac{2N^{\frac{1}{\gamma}-\frac{1}{4}}}{\mathbb{E}\{\omega_k\}\sqrt{c_2}}, c_3 N^{-\frac{15}{2\gamma}} \right]$$

is non-empty. If $N \geq N_1$ and $\kappa \in I_N$, then

$$\mathbb{P}\left(\omega \in \Omega \mid \lambda_{\kappa,\omega}(N) - 1 \leq N^{-\frac{1}{2}}\right) \leq N^{1-\frac{1}{\gamma}}\, \mathrm{e}^{-c_4 N^{1/\gamma}} \tag{3.1}$$

for a constant $c_4 > 0$ depending only on μ.

Remark 3.2 The definition of the interval I_N encodes how our weak-disorder regime depends on the length scale N. As an instance let us choose $\gamma = 35$. Then it is possible to choose $\kappa = c_3 N^{-\frac{1}{4}}$, which means that the allowed disorder regime does not shrink very fast for $N \to \infty$. For larger γ the behavior is even better.

Using a Combes-Thomas estimate we arrive at the following estimate on the probability that the Green's function, resp. resolvent, exhibits exponential off-diagonal decay for energies very close the energy one, i.e. the overall minimum of the spectrum. Note that for N large $I_N \subset [0, 1]$.

Corollary 3.3 *Let g, μ, I_N, γ, N_1 and c_4 be as in Theorem 3.1. Let $\kappa \in I_N \cap [0, 1]$, $\alpha, \beta \geq 2$ and set*

$$A := \{x \in \mathbb{R}^2 \mid 0 \leq x_1 \leq \alpha, \kappa G(x_1, \omega) < x_2 < \kappa G(x_1, \omega) + \pi\} \subset D_{\kappa,\omega}(N),$$

$$B := \{x \in \mathbb{R}^2 \mid L - \beta \leq x_1 \leq L, \kappa G(x_1, \omega) < x_2 < \kappa G(x_1, \omega) + \pi\} \subset D_{\kappa,\omega}(N).$$

Then we have for any $N \geq N_1$

$$\mathbb{P}\left(\omega \in \Omega \mid \forall\lambda \in [1, 1 + 1/(2\sqrt{N})]\colon\ \left\|\chi_A(\mathcal{H}_{\kappa,\omega}(N) - \lambda)^{-1}\chi_B\right\| \leq 2\sqrt{N}\mathrm{e}^{\left(-\frac{\operatorname{dist}(A,B)}{48\sqrt{N}}\right)}\right)$$
$$\geq 1 - N^{1-\frac{1}{\gamma}}\,\mathrm{e}^{-c_4 N^{1/\gamma}}.$$

In the typical formulation of an initial scale estimate for the multiscale analysis one requires the probability of

$$\left\{\exists\lambda \in [1, 1 + 1/(2\sqrt{N})]\colon\ \left\|\chi_A(\mathcal{H}_{\kappa,\omega}(N) - \lambda)^{-1}\chi_B\right\| > 2\sqrt{N}\mathrm{e}^{-\frac{\operatorname{dist}(A,B)}{48\sqrt{N}}}\right\}$$

to be bounded by an inverse power of N. Let $q \in \mathbb{N}$ and N_0 be such that $N^{(1+q-\frac{1}{\gamma})/c_4} \leq \exp(N^{\frac{1}{\gamma}})$ for all $N \geq N_0$. Set $N_2 := \max(N_1, N_0)$. Then for any $N \geq N_2$ we have

$$\mathbb{P}\left(\omega \in \Omega \mid \forall\lambda \in [1, 1 + 1/(2\sqrt{N})]\colon\ \left\|\chi_A(\mathcal{H}_{\kappa,\omega}(N) - \lambda)^{-1}\chi_B\right\| \leq 2\sqrt{N}\mathrm{e}^{-\left(-\frac{\operatorname{dist}(A,B)}{48\sqrt{N}}\right)}\right)$$
$$\geq 1 - N^{-q}.$$

4 Proof of Theorem 3.1 and Corollary 3.3

In this section we prove that Theorem 3.1 is implied by the following Theorem and its Corollary. We also show how Corollary 3.3 follows.

Theorem 4.1 *Recall that $(\omega_k)_{k\in\mathbb{Z}}$ is a sequence of non-negative reals. Let g, l, $\mathcal{H}_{\kappa,\omega}(N)$, $\lambda_{\kappa,\omega}(N)$ be as in Sect. 2. Recall that*

$$\tilde{g} := g - \frac{1}{l}\int_0^l g(t)\,\mathrm{d}t.$$

Assume that

$$\kappa\|\omega\|_{N,2} \leq \frac{3}{5000}\,\|\tilde{g}\|^2_{L_2(0,l)}\,\frac{1}{l^7 N^7} = \frac{c_3}{N^7}$$

Then the estimate

$$\lambda_{\kappa,\omega}(N) - 1 \geq \frac{3}{2}\,\|\tilde{g}\|^2_{L_2(0,l)}\,\frac{\kappa^2\|\omega\|^2_{N,2}}{l^3 N^3} = c_2\frac{\kappa^2\|\omega\|^2_{N,2}}{N^3}$$

holds true.

Corollary 4.2 *If $\kappa \le c_3 N^{-\frac{15}{2}}$, then we have for $\mathbb{P}$-almost all $\omega \in \Omega$:*

$$\lambda_{\kappa,\omega}(N) - 1 \ge c_2 \frac{\kappa^2 \|\omega\|_{N,2}^2}{N^3}.$$

Proof This follows immediately from Theorem 4.1, since $\|\omega\|_{N,2} \le \sqrt{N}\|\omega\|_{N,\infty} \le \sqrt{N}$ for almost all ω. Thus $\kappa \le constant\, N^{-\frac{15}{2}}$ implies $\kappa\|\omega\|_2 \le constant\, N^{-7}$. □

The following *large deviations principle* will be used in the proof of Theorem 3.1.

Lemma 4.3 *Let $\omega_k, k \in \mathbb{Z}$ be an i.i.d. sequence of non-trivial, non-negative, bounded random variables. Then there exists a constant $c_4 > 0$ depending only on μ such that*

$$\forall n \in \mathbb{N}: \quad \mathbb{P}\left(\omega \mid \frac{1}{n}\sum_{k=1}^{n} \omega_k \le \frac{\mathbb{E}\{\omega_k\}}{2}\right) \le \mathrm{e}^{-c_4 n}.$$

Proof of Theorem 3.1 For $K \in 2\mathbb{N}$ and $\gamma \in \mathbb{N}$ we set $N := K^\gamma$. Note that we can decompose a long waveguide segment into smaller parts. Thus, up to a set of measure zero $D_{\kappa,\omega}(N)$ equals

$$\dot{\bigcup_{j=0,\ldots,J-1}} D_{\kappa,\omega}(K, j)$$

where $J = N/K = K^{\gamma-1} = N^{1-\frac{1}{\gamma}}$ and $\dot{\bigcup}$ denotes a disjoint union. According to the decomposition of the segment $D_{\kappa,\omega}(N)$, we introduce new Neumann boundary conditions, which decreases the operator. More precisely, we have in the sense of quadratic forms

$$\mathcal{H}_{\kappa,\omega}(N) \ge \bigoplus_{j=0}^{J-1} \mathcal{H}_{\kappa,\omega}(K, j).$$

In particular,

$$\lambda_{\kappa,\omega}(N) \ge \min_{j=0}^{J-1} \lambda_{\kappa,\omega}(K, j). \tag{4.1}$$

The above considerations can be turned into a probabilistic estimate on the position of the lowest eigenvalue. Similar ideas have been used e.g. in [17, 30] to obtain an initial scale estimate. First note that by (4.1) we have the inclusion

$$\left\{\omega \in \Omega \mid \lambda_{\kappa,\omega}(N) - 1 \le N^{-\frac{1}{2}}\right\} \subset \bigcup_{j=0}^{J-1} \left\{\omega \in \Omega \mid \lambda_{\kappa,\omega}(K, j) - 1 \le K^{-\frac{\gamma}{2}}\right\}$$

Since the random variables $\omega_k, k \in \mathbb{Z}$ are independent and identically distributed,

$$\sum_{j=0}^{J-1} \mathbb{P}\left(\omega \in \Omega \mid \lambda_{\kappa,\omega}(K, j) - 1 \le K^{-\frac{\gamma}{2}}\right) \le N^{1-\frac{1}{\gamma}}\, \mathbb{P}\left(\omega \in \Omega \mid \lambda_{\kappa,\omega}(K) - 1 \le K^{-\frac{\gamma}{2}}\right).$$

By Corollary 4.2 and $\|\omega\|_{K,1} \leq \sqrt{K}\|\omega\|_{K,2}$ the following inclusions hold for all $\kappa \leq c_3 K^{-\frac{15}{2}}$:

$$\left\{\omega \mid \lambda_{\kappa,\omega}(K) - 1 \leq K^{-\frac{\gamma}{2}}\right\} \subset \left\{\omega \mid c_2 \frac{\kappa^2 \|\omega\|_{K,2}^2}{K^3} \leq K^{-\frac{\gamma}{2}}\right\}$$
$$= \left\{\omega \mid \|\omega\|_{K,2} \leq \frac{1}{\kappa\sqrt{c_2}} K^{\frac{3-\gamma/2}{2}}\right\} \subset \left\{\omega \mid \frac{\|\omega\|_{K,1}}{K} \leq \frac{1}{\kappa\sqrt{c_2}} K^{1-\frac{\gamma}{4}}\right\}.$$

Denote by $\mathbb{E}\{\omega_k\}$ the expectation value of (any) ω_k, and choose now κ such that

$$\frac{K^{1-\frac{\gamma}{4}}}{\kappa\sqrt{c_2}} \leq \frac{\mathbb{E}\{\omega_k\}}{2} \quad \text{i.e.} \quad \frac{2K^{1-\frac{\gamma}{4}}}{\mathbb{E}\{\omega_k\}\sqrt{c_2}} \leq \kappa. \tag{4.2}$$

The upper and the lower bound for κ can be reconciled if $\gamma > 34$ and

$$K \geq K_1 := \left(\frac{2}{\mathbb{E}\{\omega_k\} c_3 \sqrt{c_2}}\right)^{\frac{2}{\gamma-34}}.$$

The last inequality is equivalent to $N \geq N_1 := (\mathbb{E}\{\omega_k\} c_3 \sqrt{c_2}/2)^{\frac{-2\gamma}{\gamma-34}}$. For κ satisfying (4.2) we are able to apply the large deviations principle of Lemma 4.3 and thus obtain

$$\mathbb{P}\left(\omega \mid \frac{\|\omega\|_1}{K} \leq \frac{1}{\kappa\sqrt{c_2}} K^{1-\frac{\gamma}{4}}\right) \leq \mathbb{P}\left(\omega \mid \frac{\|\omega\|_1}{K} \leq \frac{\mathbb{E}\{\omega_k\}}{2}\right) \leq \mathrm{e}^{-c_4 K}$$

for $K \geq K_1$. Consequently we have for $N \geq N_1$

$$\mathbb{P}\left(\omega \in \Omega \mid \lambda_{\kappa,\omega}(N) - 1 \leq N^{-\frac{1}{2}}\right) \leq N^{1-\frac{1}{\gamma}} \mathrm{e}^{-c_4 N^{1/\gamma}}.$$

□

Proof of Corollary 3.3 Since $\lambda_{\kappa,\omega}(N)$ is the element of $\sigma(\mathcal{H}_{\kappa,\omega}(N))$ closest to one,

$$\Omega' = \left\{\omega \in \Omega \mid \lambda_{\kappa,\omega}(N) - 1 > N^{-\frac{1}{2}}\right\} = \left\{\omega \in \Omega \mid \mathrm{dist}(1, \sigma(\mathcal{H}_{\kappa,\omega}(N)) > N^{-\frac{1}{2}}\right\}$$
$$= \left\{\omega \in \Omega \mid \forall\lambda \in [1, 1 + 1/(2\sqrt{N})]\colon \mathrm{dist}(\lambda, \sigma(\mathcal{H}_{\kappa,\omega}(N)) > 1/(2\sqrt{N})\right\}.$$

For $\omega \in \Omega'$ we want to use a Combes-Thomas argument to obtain a decay estimate for the Green's function from the estimate on the distance of the relevant energies to the spectrum. Such estimates are rather standard for Schrödinger-type operators. However, we did not find a specific formulation of this result in the literature which fits exactly our situation, so we provide a direct proof, for the convenience of the reader.

In the following the parameters $\omega \in \Omega'$, $\kappa > 0$, and $N \in \mathbb{N}$ will be kept fixed. For this reason we shall abbreviate in the subsequent calculations $\mathcal{H}_{\kappa,\omega}(N)$ simply by $\mathcal{H}$. Define the function $J\colon [0, l] \to [0, \infty)$ by setting $J(t) = t$ for $t \in (1, L-1)$, $J(t) = 3t^2 - 3t^3 + t^4$ for $t \in [0, 1]$, and choosing the values on the segment $[L-1, L]$ in such a way that the graph of the function becomes point symmetric w.r.t. $(L/2, L/2)$. Note that J satisfies Neumann b.c. at $t = 0$ and $t = L$, that it is twice differentiable, and that on the segment $t \in [0, 1]$

$$J'(t) = 6t - 9t^2 + 4t^3 \geq 0, \qquad J''(t) = 6 - 18t + 12t^2.$$

It thus follows that

$$\|J'\|_\infty \le \frac{5}{4}, \qquad \|J''\|_\infty \le 6,$$

and that J is monotonously increasing. For $a \in (0,1)$ we define the multiplication operator

$$\mathcal{T}_a : L_2(D_{\kappa,\omega}(N)) \to L_2(D_{\kappa,\omega}(N)), \qquad (\mathcal{T}_a f)(x) := \mathrm{e}^{aJ(x_1)} f(x)$$

and another operator

$$\mathcal{P}_a := -2aJ' \frac{\partial}{\partial x_1} - a^2 J' - aJ'',$$

which will turn out to be an 'effective perturbation'. A direct calculation shows that

$$\mathcal{T}_{-a}\mathcal{T}_a = \mathrm{I}, \qquad \mathcal{T}_{-a}\mathcal{H}\mathcal{T}_a = \mathcal{H} + \mathcal{P}_a, \qquad \mathcal{T}_{-a}(\mathcal{H}-\lambda)^{-1}\mathcal{T}_a = (\mathcal{H}+\mathcal{P}_a-\lambda)^{-1},$$

provided that λ is in the intersection of the resolvent sets of the two operators $\mathcal{H}$ and $\mathcal{H}+\mathcal{P}_a$. We shall identify a range of values for a such that the last condition holds and we get even an explicit bound on the norm of $(\mathcal{H}+\mathcal{P}_a-\lambda)^{-1}$. In these considerations we shall keep in mind that λ is close to, but larger that one, in particular $\lambda \in [1,2]$. We denote by $\delta = \mathrm{dist}(\sigma(\mathcal{H}), \lambda)$ the distance of the spectrum of $\mathcal{H}$ to λ. Let us first estimate

$$\begin{aligned}\|\mathcal{P}_a(\mathcal{H}-\lambda)^{-1}\| &\le a\left[(a\|J'\|_\infty^2 + \|J''\|_\infty)\|(\mathcal{H}-\lambda)^{-1}\| + 2\|J'\|_\infty \left\|\frac{\partial}{\partial x_1}(\mathcal{H}-\lambda)^{-1}\right\|\right] \\ &\le a\left[\frac{\frac{25}{16}a + 6}{\delta} + \frac{5}{2}\sqrt{\frac{\lambda}{\delta^2} + \frac{1}{\delta}}\right]. \end{aligned} \tag{4.3}$$

Here we have employed that

$$\left\|\frac{\partial}{\partial x_1}(\mathcal{H}-\lambda)^{-1}\right\| \le \sqrt{\frac{\lambda}{\delta^2} + \frac{1}{\delta}},$$

which follows directly from the obvious relations

$$\|\nabla u\|^2_{L_2(D_{\kappa,\omega})} - \lambda\|u\|^2_{L_2(D_{\kappa,\omega})} = (f,u)_{L_2(D_{\kappa,\omega})}, \qquad \|u\|_{L_2(D_{\kappa,\omega})} \le \frac{1}{\delta}\|f\|_{L_2(D_{\kappa,\omega})},$$

where $u := (\mathcal{H}-\lambda)^{-1} f$, $f \in L_2(D_{\kappa,\omega})$.

The right hand side in (4.3) is bounded by

$$a\left[\frac{121}{16\delta} + \frac{5}{2}\sqrt{\frac{\lambda+\delta}{\delta^2}}\right] \le 12\frac{a}{\delta},$$

since $\delta, a \in [0,1]$ and $\lambda \in [1,2]$. Now choose $a = \frac{\delta}{24} \le \frac{1}{2}\frac{\delta}{12}$. Then $\|\mathcal{P}_a(\mathcal{H}-\lambda)^{-1}\| \le 1/2$ and thus the norm of the Neumann series satisfies

$$\|(\mathcal{H}+\mathcal{P}_a-\lambda)^{-1}\| \le \frac{\|(\mathcal{H}-\lambda)^{-1}\|}{1-\frac{1}{2}} = \frac{2}{\delta}.$$

Now choose $\alpha, \beta \ge 2$ and set

$$A := \{x \in \mathbb{R}^2 \mid 0 \le x_1 \le \alpha, \kappa G(x_1,\omega) < x_2 < \kappa G(x_1,\omega) + \pi\} \subset D_{\kappa,\omega}(N),$$

$$B := \{x \in \mathbb{R}^2 \mid L - \beta \le x_1 \le L, \kappa G(x_1, \omega) < x_2 < \kappa G(x_1, \omega) + \pi\} \subset D_{\kappa,\omega}(N).$$

For any normalised vectors $\phi, \psi \in L_2(D_{\kappa,\omega}(N))$ we have

$$\left|\langle |\psi| \chi_A, \mathcal{T}_{-a}(\mathcal{H} - \lambda)^{-1} \mathcal{T}_a \chi_B |\phi| \rangle\right| \le \|\mathcal{T}_{-a}(\mathcal{H} - \lambda)^{-1} \mathcal{T}_a\| = \|(\mathcal{H} + \mathcal{P}_a - \lambda)^{-1}\| \le \frac{2}{\delta}.$$

Due to the monotonicity of J and the positivity of the integral kernel of $(\mathcal{H} - \lambda)^{-1}$ we are able to estimate

$$\left|\langle |\psi| \chi_A, \mathcal{T}_{-a}(\mathcal{H} - \lambda)^{-1} \mathcal{T}_a \chi_B |\phi| \rangle\right| \ge \mathrm{e}^{a(J(L-\beta)-J(\alpha))} \left|\langle \psi \chi_A, (\mathcal{H} - \lambda)^{-1} \chi_B \phi \rangle\right|.$$

Note that by the choice of the values of α, β and the function J we have $J(L - \beta) - J(\alpha) = L - \beta - \alpha = \mathrm{dist}(A, B)$. Bringing the exponential term on the other side we obtain

$$\left|\langle \psi \chi_A, (\mathcal{H} - \lambda)^{-1} \chi_B \phi \rangle\right| \le \frac{2}{\delta} \mathrm{e}^{-a\, \mathrm{dist}(A,B)} = \frac{2}{\delta} \mathrm{e}^{-\frac{\mathrm{dist}(A,B)\delta}{24}}$$

Now fix $\omega \in \Omega' = \{\omega \in \Omega \mid \lambda_{\kappa,\omega}(N) - 1 > N^{-\frac{1}{2}}\}$ and $\lambda \in [1, 1 + \frac{1}{2\sqrt{N}}]$. Then $\delta \ge \frac{1}{2\sqrt{N}}$ and thus

$$\left|\langle \psi \chi_A, (\mathcal{H} - \lambda)^{-1} \chi_B \phi \rangle\right| \le \frac{2}{\delta} \mathrm{e}^{-\frac{\mathrm{dist}(A,B)\delta}{24}} \le 2\sqrt{N} \mathrm{e}^{-\frac{\mathrm{dist}(A,B)}{48\sqrt{N}}}.$$

Since we have by the estimate (3.1) the bound $\mathbb{P}(\Omega') \ge 1 - N^{1-\frac{1}{\gamma}}\, \mathrm{e}^{-c_4 N^{1/\gamma}}$ for $N \ge N_1$, we conclude that

$$\mathbb{P}\left(\omega \in \Omega \mid \forall \lambda \in [1, 1 + 1/(2\sqrt{N})] \colon \left\| \chi_A (\mathcal{H}_{\kappa,\omega}(N) - \lambda)^{-1} \chi_B \right\| \le 2\sqrt{N} \mathrm{e}^{-\frac{\mathrm{dist}(A,B)}{48\sqrt{N}}}\right)$$
$$\ge 1 - N^{1-\frac{1}{\gamma}}\, \mathrm{e}^{-c_4 N^{1/\gamma}}. \tag{4.4}$$

□

5 Deterministic Lower Bounds

In this section only finite segments of the waveguide will be relevant to us. Recall that due to the fact that the support of the measure μ is contained in the interval [0, 1], for almost all $\omega \in \Omega$ the bound $\|\omega\|_\infty \le 1$ holds. In this section an ℓ^2-normalisation will be better suited, and in fact possible since only finite waveguide segments, and thus only finite families of random variables $\omega_1, \dots, \omega_N$ are involved.

We fix $N \in \mathbb{N}$ and set for all $j \in \{1, \dots, N\}$

$$\rho_j := \kappa\, \omega_j, \theta_j := \rho_j / \varepsilon, \qquad \text{where} \quad \varepsilon := \left(\sum_{j=1}^N \rho_j^2\right)^{\frac{1}{2}} = \kappa \|\omega\|_{N,2}$$

Observe that the vector $\theta = \{\theta_i\}_{i=1}^N$ is normalized in the sense

$$\theta_1^2 + \dots + \theta_N^2 = 1. \tag{5.1}$$

Let $l \geq 1$, $g\colon \mathbb{R} \to \mathbb{R}$, $\tilde{g} \in \mathbb{R}$, $G\colon \mathbb{R} \times \Omega \to \mathbb{R}$ be as in Sects. 2 and 3. We set $L := Nl$ and consider in the following the restriction of G to the set $[0, L] \times \big(\times_{j=1}^{N} [0, \infty)\big)$. The restricted function will be again denoted by G, and the finite vector $(\omega_1, \ldots, \omega_N)$ will be denoted by ω, by slight abuse of notation. In terms of the new parameters we have

$$\kappa G(x_1, \omega) = G(x_1, \kappa\omega) = G(x_1, \rho) = G(x_1, \varepsilon\theta) = \varepsilon G(x_1, \theta) = \varepsilon \sum_{j=0}^{N} \theta_j g(x_1 - jl).$$

For the subset of $\mathbb{R}^2$ forming the finite segment of the waveguide we shall use in this section the notation

$$\Pi_\rho := \{x \in \mathbb{R}^2 \mid 0 < x_1 < L, \varepsilon G(x_1, \theta) < x_2 < \varepsilon G(x_1, \theta) + \pi\}$$

Let us point out that $\Pi_\rho = D_{\|\rho\|_\infty, \rho/\|\rho\|_\infty}(N)$ in the notation of Sect. 2 and that $\Pi_0 = \{x \in \mathbb{R}^2 \mid 0 < x_1 < L, 0 < x_2 < \pi\}$. Similarly as before

$$\Gamma_\rho := \{x \in \mathbb{R}^2 \mid x_1 \in (0, L), x_2 = \varepsilon G(x_1, \theta)\} \cup \{x : x_1 \in (0, L), x_2 = \varepsilon G(x_1, \theta) + \pi\}$$

indicates the upper and lower parts of the boundary of Π_ρ and the remainder of the boundary $\partial\Pi_\rho \setminus \overline{\Gamma}_\rho$ is denoted by γ_ρ. By $\mathcal{H}_\rho$ we denote the negative Laplacian in Π_ρ with Dirichlet boundary conditions on Γ_ρ and Neumann ones on γ_ρ and by $\lambda(\rho)$ the lowest eigenvalue of $\mathcal{H}_\rho$.

In what follows $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the scalar product in $L_2(\Pi_0)$ and the associated norm. We repeat here the statement of Theorem 4.1 in the new parametrization.

Theorem 5.1 *For*

$$\varepsilon \leq \frac{3}{5000} \|\tilde{g}\|^2_{L_2(0,l)} \frac{1}{L^7} \tag{5.2}$$

the estimate

$$\lambda(\rho) - 1 \geq \frac{3}{2} \|\tilde{g}\|^2_{L_2(0,l)} \frac{\varepsilon^2}{L^3}$$

holds true.

The rest of the section is devoted to the proof of the theorem.

First we transform the Laplacian on the wiggled strip segments into a differential operator with variable coefficients on a rectangle. It is then possible to treat the latter operator as a perturbation of the pure Laplacian on the rectangle. We introduce the coordinates $\xi := (\xi_1, \xi_2)$, $\xi_1 := x_1$, $\xi_2 := x_2 - \varepsilon G(x_1)$. The mapping $u(x) \mapsto u(\xi_1, \xi_2 + \varepsilon G(\xi_1))$ is a unitary operator from $L_2(\Pi_\rho)$ to $L_2(\Pi_0)$. Let ψ_ρ be the normalized eigenfunction associated with $\lambda(\rho)$. It is the unique solution of the boundary value problem

$$(-\Delta_\xi - \varepsilon \mathcal{Q}_\rho)\psi_\rho = \lambda(\rho)\psi_\rho, \quad \xi \in \Pi_0, \tag{5.3}$$

$$\psi_0 = 0, \quad x \in \Gamma_0, \qquad \frac{\partial \psi_0}{\partial \xi_1} = 0, \quad \xi \in \gamma_0, \tag{5.4}$$

where

$$\mathcal{Q}_\rho := -2G' \frac{\partial^2}{\partial\xi_1 \partial\xi_2} + \varepsilon\big(G'\big)^2 \frac{\partial^2}{\partial\xi_2^2} - G'' \frac{\partial}{\partial\xi_2}.$$

Hereafter the prime denotes the derivative w.r.t. ξ_1.

For the perturbation theoretic estimates we are aiming for we need some control of the resolvent set. The first two eigenvalues of $\mathcal{H}_0$ are 1 and $1+\pi^2/L^2$. The eigenfunction corresponding to the eigenvalue one is

$$\psi_0 \colon \Pi_0 \to \Pi_0, \qquad \psi_0(\xi) := \sqrt{\frac{2}{\pi L}} \sin \xi_2.$$

It turns out that it is more convenient to work with the modified resolvent

$$\mathcal{R}_0(\lambda) := (\mathcal{H}_0 - \lambda)^{-1} - \frac{\langle \cdot, \psi_0 \rangle}{1-\lambda} \psi_0.$$

which is a bounded self-adjoint operator for any

$$\lambda \in B_{\pi^2/(2L^2)}(1) \subset \mathbb{C}. \tag{5.5}$$

The operator-valued function $B_{\pi^2/(2L^2)}(1) \ni \lambda \mapsto \mathcal{R}_0(\lambda)$ is holomorphic, cf. [14, Chap. V, Sect. 3.5].

For an arbitrary $f \in L_2(\Pi_0)$ set $\hat{f} := f - \langle f, \psi_0\rangle \psi_0$. Then the function $u := \mathcal{R}_0(\lambda) f = (\mathcal{H}_0 - \lambda)^{-1} \hat{f}$ solves the equation

$$(\mathcal{H}_0 - \lambda) u = (\mathcal{H}_0 - \lambda)\,(\mathcal{H}_0 - \lambda)^{-1} f + (\mathcal{H}_0 - \lambda) \frac{\langle f, \psi_0\rangle}{\lambda - 1} \psi_0 = \hat{f}, \tag{5.6}$$

and is orthogonal to ψ_0 in $L_2(\Pi_0)$.

Lemma 5.2 *For $f \in L_2(\Pi_0)$, $u := \mathcal{R}_0(\lambda) f$ and $\lambda \in B_{\pi^2/(2L^2)}(1) \subset \mathbb{C}$ we have $u \in W_2^2(\Pi_0)$ and*

$$\begin{aligned} \|u\| &\le \frac{2L^2}{\pi^2}\|f\|, & \|\nabla u\| &\le \frac{7L^2}{\pi^2}\|f\|, \\ \left\|\nabla \frac{\partial u}{\partial \xi_1}\right\| &\le \frac{25L^2}{\pi^2}\|f\|, & \left\|\frac{\partial^2 u}{\partial \xi_2^2}\right\| &\le \frac{47L^2}{\pi^2}\|f\|. \end{aligned} \tag{5.7}$$

Proof The vector $u = (\mathcal{H}_0 - \lambda)^{-1} \hat{f}$ is in the range of $(\mathcal{H}_0 - \lambda)^{-1}$ and thus in the Sobolev space $W_2^2(\Pi_0)$. The first inequality follows from [14, Chap. V] and the fact that our choice of λ is separated by at least $\pi^2/(2L^2)$ from the next spectral value.

Let us prove the second estimate. We begin with the obvious identity

$$\|\hat{f}\|^2 = \|f\|^2 - |\langle f, \psi_0\rangle^2| \le \|f\|^2. \tag{5.8}$$

We multiply (5.6) with u and obtain $\|\nabla u\|^2 = \lambda \|u\|^2 + \langle \hat{f}, u\rangle$. This identity, (5.5) and the first estimate in (5.7) yield

$$\|\nabla u\|^2 \le |\lambda| \|u\|^2 + \|f\|\,\|u\| \le \frac{4L^4}{\pi^4}|\lambda| \|f\|^2 + \frac{2L^2}{\pi^2}\|f\|^2$$

$$\leq \frac{4L^4}{\pi^4}\left(1+\frac{\pi^2}{2L^2}+\frac{\pi^2}{2L^2}\right)\|f\|^2 \leq \frac{4L^4}{\pi^4}\left(1+\pi^2\right)\|f\|^2 \tag{5.9}$$

that proves the desired estimate.

Since $u \in W_2^1(\Pi_0)$, the function $v := \frac{\partial u}{\partial \xi_1}$ is a generalized solution to the boundary value problem

$$-\Delta_\xi v = \lambda v + \frac{\partial \hat{f}}{\partial \xi_1}, \quad \xi \in \Pi_0, \qquad v = 0, \quad \xi \in \partial\Pi_0$$

which is obtained by differentiating equation (5.6). We multiply the last equation by v, integrate by parts, and obtain

$$\|\nabla v\|^2 = \lambda\|v\|^2 - \left\langle \hat{f}, \frac{\partial v}{\partial \xi_1}\right\rangle.$$

We employ (5.5), (5.8), and Young's inequality to estimate

$$\|\nabla v\|^2 \leq \left(1+\frac{\pi^2}{2L^2}\right)\|v\|^2 + \frac{1}{2}\|f\|^2 + \frac{1}{2}\left\|\frac{\partial v}{\partial \xi_1}\right\|^2 \leq \left(1+\frac{\pi^2}{2L^2}\right)\|\nabla u\|^2 + \frac{1}{2}\|f\|^2 + \frac{1}{2}\|\nabla v\|^2,$$

and hence by (5.9)

$$\begin{aligned}
\left\|\nabla \frac{\partial u}{\partial \xi_1}\right\|^2 &= \|\nabla v\|^2 \leq \left(2+\frac{\pi^2}{L^2}\right)\|\nabla u\|^2 + \|f\|^2 \\
&\leq \left(4(\pi^2+1)\left(2+\frac{\pi^2}{L^2}\right)+\frac{\pi^4}{L^4}\right)\frac{L^4}{\pi^4}\|f\|^2 \\
&\leq \left(4(\pi^2+1)(2+\pi^2)+\pi^4\right)\frac{L^4}{\pi^4}\|f\|^2 \leq \frac{614L^4}{\pi^4}\|f\|^2.
\end{aligned} \tag{5.10}$$

This implies the penultimate inequality in (5.7).

We rewrite (5.6) as

$$-\frac{\partial^2 u}{\partial \xi_2^2} = \frac{\partial^2 u}{\partial \xi_1^2} + \lambda u + \hat{f},$$

and see that the estimates (5.5), (5.8), (5.10), and the first estimate in (5.7) yield

$$\begin{aligned}
\left\|\frac{\partial^2 u}{\partial \xi_2^2}\right\| &\leq \left\|\frac{\partial^2 u}{\partial \xi_1^2}\right\| + |\lambda|\,\|u\| + \|f\| \\
&\leq \left(\sqrt{614} + 2\left(1+\frac{\pi^2}{2L^2}\right)+\pi^2\right)\frac{L^2}{\pi^2}\|f\| \\
&\leq \left(\sqrt{614}+2(1+\pi^2)+\pi^2\right)\frac{L^2}{\pi^2}\|f\| \leq \frac{47L^2}{\pi^2}\|f\|,
\end{aligned}$$

which proves the last claim in (5.7). □

The inequalities in (5.2) can be understood as bounds on the norms of certain operators products:

$$\|\mathcal{R}_0(\lambda)\| \le \frac{2L^2}{\pi^2}, \qquad \|\nabla\mathcal{R}_0(\lambda)\| \le \frac{7L^2}{\pi^2},$$
$$\left\|\nabla\frac{\partial}{\partial\xi_1}\mathcal{R}_0(\lambda)\right\| \le \frac{25L^2}{\pi^2}, \qquad \left\|\frac{\partial^2}{\partial\xi_2^2}\mathcal{R}_0(\lambda)\right\| \le \frac{47L^2}{\pi^2}. \tag{5.11}$$

By the normalization (2.1) we have

$$\|\tilde{g}\|^2_{L_2(0,l)} = \|g\|^2_{L_2(0,l)} - l^{-1}\left(\int_0^l g(t)\,\mathrm{d}t\right)^2 \le \|g\|^2_{L_2(0,l)} \le l. \tag{5.12}$$

This implies together with inequality (5.2)

$$\varepsilon \le \frac{3l^2}{5000L^7} \le \frac{3}{5000}\frac{1}{L^5N^2} \le \frac{3}{5000}. \tag{5.13}$$

We use (5.1) and (2.1) and Cauchy-Schwartz inequality to establish

$$\|\mathcal{Q}_\rho u\| \le 2\left\|\frac{\partial^2 u}{\partial\xi_1\partial\xi_2}\right\| + \left\|\frac{\partial^2 u}{\partial\xi_2^2}\right\| + \left\|\frac{\partial u}{\partial\xi_2}\right\|$$

for any $u \in W_2^2(\Pi_0)$. For λ satisfying (5.5), and ε as in (5.13), Lemma 5.2 and the last bound imply

$$\|\mathcal{Q}_\rho\mathcal{R}_0(\lambda)\| \le \frac{104L^2}{\pi^2}, \qquad \varepsilon\|\mathcal{Q}_\rho\mathcal{R}_0(\lambda)\| \le \frac{8}{125\pi^2} < 1. \tag{5.14}$$

Here the norm is understood as the norm of an operator in $L_2(\Pi_0)$.

Let us show that $\lambda(\rho)$ satisfies (5.5). Indeed,

$$\lambda(\rho) = \left\|\frac{\partial\psi_\rho}{\partial\xi_1} - \varepsilon G'\frac{\partial\psi_\rho}{\partial\xi_2}\right\|^2 + \left\|\frac{\partial\psi_\rho}{\partial\xi_2}\right\|^2 \ge \left\|\frac{\partial\psi_\rho}{\partial\xi_2}\right\|^2 \ge 1. \tag{5.15}$$

By the minimax principle and (5.1), (2.1) the eigenvalue $\lambda(\rho)$ can be estimated from above as follows

$$\lambda(\rho) \le \left\|\frac{\partial\psi_0}{\partial\xi_1} - \varepsilon G'\frac{\partial\psi_0}{\partial\xi_2}\right\|^2 + \left\|\frac{\partial\psi_0}{\partial\xi_2}\right\|^2 \le 1 + \varepsilon^2. \tag{5.16}$$

Two last estimates and (5.13) imply (5.5) for $\lambda(\rho)$.

Next we shall do some perturbation theory for linear operators to be able to identify the dominating contributions. Using the proved fact, (5.14), and proceeding completely as in [9], [3, Sect. 4], one can show that for the considered values of ε the eigenvalue $\lambda(\rho)$ solves the equation

$$\lambda(\rho) - 1 = -\varepsilon\big\langle(\mathrm{I} - \varepsilon\mathcal{Q}_\rho\mathcal{R}_0(\lambda(\rho)))^{-1}\mathcal{Q}_\rho\psi_0, \psi_0\big\rangle, \tag{5.17}$$

where I is the identity mapping. A direct calculation shows

$$(\mathrm{I} - \varepsilon\mathcal{Q}_\rho\mathcal{R}_0(\lambda(\rho)))^{-1} = \mathrm{I} + \varepsilon\mathcal{Q}_\rho\mathcal{R}_0(\lambda(\rho)) + \varepsilon^2(\mathrm{I} - \varepsilon\mathcal{Q}_\rho\mathcal{R}_0(\lambda(\rho)))^{-1}\big(\mathcal{Q}_\rho\mathcal{R}_0(\lambda(\rho))\big)^2.$$

We substitute this relation into (5.17),

$$\lambda(\rho) - 1 = -\varepsilon\langle\mathcal{Q}_\rho\psi_0, \psi_0\rangle - \varepsilon^2\langle\mathcal{Q}_\rho\mathcal{R}_0(\lambda(\rho))\mathcal{Q}_\rho\psi_0, \psi_0\rangle$$

$$-\varepsilon^3\Big\langle(\mathrm{I}-\varepsilon\mathcal{Q}_\rho\mathcal{R}_0(\lambda(\rho)))^{-1}\big(\mathcal{Q}_\rho\mathcal{R}_0(\lambda(\rho))\big)^2\mathcal{Q}_\rho\psi_0,\psi_0\Big\rangle. \tag{5.18}$$

We first consider the coefficient of ε^2. For the sake of brevity set

$$v:=\mathcal{R}_0(\lambda(\rho))\mathcal{Q}_\rho\psi_0.$$

Integrating by parts, we obtain

$$\begin{aligned}
\langle\mathcal{Q}_\rho\mathcal{R}_0(\lambda(\rho))\mathcal{Q}_\rho\psi_0,\psi_0\rangle =&\langle\mathcal{Q}_\rho v,\psi_0\rangle\\
=&-2\Big\langle G'\frac{\partial^2 v}{\partial\xi_1\partial\xi_2},\psi_0\Big\rangle-\Big\langle G''\frac{\partial v}{\partial\xi_2},\psi_0\Big\rangle+\varepsilon\Big\langle(G')^2\frac{\partial^2 v}{\partial\xi_2^2},\psi_0\Big\rangle\\
=&-2\Big\langle G''v,\frac{\partial\psi_0}{\partial\xi_2}\Big\rangle+\Big\langle G''v,\frac{\partial\psi_0}{\partial\xi_2}\Big\rangle+\varepsilon\Big\langle(G')^2v,\frac{\partial^2\psi_0}{\partial\xi_2^2}\Big\rangle\\
=&-\Big\langle G''v,\frac{\partial\psi_0}{\partial\xi_2}\Big\rangle-\varepsilon\Big\langle(G')^2v,\psi_0\Big\rangle\\
=&-\Big\langle G''\mathcal{R}_0(\lambda(\rho))\mathcal{Q}_\rho\psi_0,\frac{\partial\psi_0}{\partial\xi_2}\Big\rangle-\varepsilon\langle(G')^2\mathcal{R}_0(\lambda(\rho))\mathcal{Q}_\rho\psi_0,\psi_0\rangle
\end{aligned}$$

which equals

$$\begin{aligned}
&\Big\langle G''\mathcal{R}_0(\lambda(\rho))G''\frac{\partial\psi_0}{\partial\xi_2},\frac{\partial\psi_0}{\partial\xi_2}\Big\rangle+\varepsilon\Big\langle G''\mathcal{R}_0(\lambda(\rho))(G')^2\psi_0,\frac{\partial\psi_0}{\partial\xi_2}\Big\rangle\\
&-\varepsilon\langle(G')^2\mathcal{R}_0(\lambda(\rho))\mathcal{Q}_\rho\psi_0,\psi_0\rangle.
\end{aligned}$$

Since $\mathcal{R}_0(\lambda)$ is a (special) rank one perturbation of $(\mathcal{H}_0-\lambda)^{-1}$, cf. (5.6), it inherits the resolvent identity

$$\mathcal{R}_0(\lambda)-\mathcal{R}_0(\mu)=(\lambda-\mu)\mathcal{R}_0(\mu)\mathcal{R}_0(\lambda).$$

Now we substitute two last identities into (5.18),

$$\begin{aligned}
\lambda(\rho)-1=&-\varepsilon\langle\mathcal{Q}_\rho\psi_0,\psi_0\rangle-\varepsilon^2\Big\langle G''\mathcal{R}_0(1)G''\frac{\partial\psi_0}{\partial\xi_2},\frac{\partial\psi_0}{\partial\xi_2}\Big\rangle\\
&+\varepsilon^3\langle(G')^2\mathcal{R}_0(\lambda(\rho))\mathcal{Q}_\rho\psi_0,\psi_0\rangle\\
&-\varepsilon^3\Big\langle G''\mathcal{R}_0(\lambda(\rho))(G')^2\psi_0,\frac{\partial\psi_0}{\partial\xi_2}\Big\rangle\\
&-\varepsilon^2(\lambda(\rho)-1)\Big\langle G''\mathcal{R}_0(1)\mathcal{R}_0(\lambda(\rho))G''\frac{\partial\psi_0}{\partial\xi_2},\frac{\partial\psi_0}{\partial\xi_2}\Big\rangle\\
&-\varepsilon^3\Big\langle(\mathrm{I}-\varepsilon\mathcal{Q}_\rho\mathcal{R}_0(\lambda(\rho)))^{-1}\big(\mathcal{Q}_\rho\mathcal{R}_0(\lambda(\rho))\big)^2\mathcal{Q}_\rho\psi_0,\psi_0\Big\rangle.
\end{aligned} \tag{5.19}$$

It will turn out that the two first terms on the right hand side of this identity are positive and dominate all the remaining terms. The first term is of the order ε^2/L. However, a cancelation occurs such that the leading contribution turns out to be of order ε^2/L^3. A direct calculation shows that

$$\langle\mathcal{Q}_\rho\psi_0,\psi_0\rangle=-\frac{\varepsilon}{L}\|g'\|^2_{L_2(0,l)}. \tag{5.20}$$

It is more complicated to evaluate the second term. We denote

$$w := \mathcal{R}_0(1) G'' \frac{\partial \psi_0}{\partial \xi_2},$$

Since this function solves the equation

$$(\mathcal{H}_0 - 1) w = G'' \frac{\partial \psi_0}{\partial \xi_2},$$

it is natural to consider the auxiliary function

$$v := w + G \frac{\partial \psi_0}{\partial \xi_2}. \tag{5.21}$$

Using the normalization (5.1), separation of variables and

$$\left\| \frac{\partial \psi_0}{\partial \xi_2} \right\|^2_{L_2(0,\pi)} = 1/L,$$

we get

$$\left\langle G'' \mathcal{R}_0(1) G'' \frac{\partial \psi_0}{\partial \xi_2}, \frac{\partial \psi_0}{\partial \xi_2} \right\rangle = \left\langle G'' w, \frac{\partial \psi_0}{\partial \xi_2} \right\rangle = \frac{1}{L} \|g'\|^2_{L_2(0,l)} + \left\langle v, G'' \frac{\partial \psi_0}{\partial \xi_2} \right\rangle. \tag{5.22}$$

The first term on the right side leads to a cancelation of (5.20). It is clear that the function v solves the boundary value problem

$$-\Delta v = v, \quad \xi \in \Pi_0, \qquad \frac{\partial v}{\partial \xi_1} = 0, \quad \xi \in \gamma_0,$$

$$v = G \sqrt{\frac{2}{\pi L}}, \quad \xi_1 \in (0, L),\ \xi_2 = 0, \qquad v = -G \sqrt{\frac{2}{\pi L}}, \quad \xi_1 \in (0, L),\ \xi_2 = \pi.$$

We multiply the equation in this problem by $G \frac{\partial \psi_0}{\partial \xi_2}$ and integrate twice by parts. Due to the separation of variables a direct calculation shows that part of the boundary contributions vanish and the equation simplifies to

$$\left\langle v, G \frac{\partial \psi_0}{\partial \xi_2} \right\rangle = \sqrt{\frac{2}{\pi L}} \int_0^L G(\xi_1, \theta) \left(\frac{\partial v}{\partial \xi_2}\Big|_{\xi_2=\pi} + \frac{\partial v}{\partial \xi_2}\Big|_{\xi_2=0} \right) \mathrm{d}\xi_1$$
$$- \left\langle v, G'' \frac{\partial \psi_0}{\partial \xi_2} \right\rangle + \left\langle v, G \frac{\partial \psi_0}{\partial \xi_2} \right\rangle.$$

Thus

$$\left\langle v, G'' \frac{\partial \psi_0}{\partial \xi_2} \right\rangle = \sqrt{\frac{2}{\pi L}} \int_0^L G(\xi_1, \theta) \left(\frac{\partial v}{\partial \xi_2}\Big|_{\xi_2=\pi} + \frac{\partial v}{\partial \xi_2}\Big|_{\xi_2=0} \right) \mathrm{d}\xi_1. \tag{5.23}$$

We shall now expand the functions in terms of the eigenvectors of the transversal Laplacian with Dirichlet b.c. and the longitudinal Laplacian with Neumann b.c. It follows from the equation $(\mathcal{H}_0 - \lambda) w = G'' \frac{\partial \psi_0}{\partial \xi_2}$ that

$$w(\xi, \rho) = -8\pi \sqrt{\frac{2}{\pi L}} \sum_{n,m=1}^{\infty} \frac{n^2 m G_n(\rho)}{4m^2 - 1} \frac{\cos \pi n L^{-1} \xi_1 \sin 2m\xi_2}{(4m^2 - 1) L^2 + \pi^2 n^2},$$

where the coefficients $G_n(\rho)$ are defined by the identity

$$G(\xi_1,\theta)=\sum_{n=0}^{\infty}G_n(\rho)\cos\pi nL^{-1}\xi_1.$$

We substitute the expansion obtained and (5.21) into (5.23),

$$\begin{aligned}-\Big\langle v,G''\frac{\partial\psi_0}{\partial\xi_2}\Big\rangle&=32\sum_{n,m=1}^{\infty}\frac{n^2m^2G_n^2(\rho)}{4m^2-1}\frac{1}{(4m^2-1)L^2+\pi^2n^2}\\&>\frac{32}{3}\sum_{n=1}^{\infty}\frac{n^2G_n^2(\rho)}{3L^2+\pi^2n^2}>\frac{32}{3(3+\pi^2)L^2}\sum_{n=1}^{\infty}G_n^2(\rho)>\frac{64}{(9+3\pi^2)L^2}\|\tilde g\|^2_{L_2(0,l)}.\end{aligned}$$

Here we have also employed that due to (5.12) and the normalization (5.20)

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{LG_n^2(\rho)}{2}&=\|G-G_0\|^2_{L_2(0,L)}=\|G\|^2_{L_2(0,L)}-LG_0^2\\&=\|g\|^2_{L_2(0,l)}-\frac{1}{L}\left(\sum_{i=1}^{N}\theta_i\right)^2\left(\int_0^l g(t)\,\mathrm{d}t\right)^2\\&\geq\|g\|^2_{L_2(0,l)}-\frac{1}{l}\left(\int_0^l g(t)\,\mathrm{d}t\right)^2=\|\tilde g\|^2_{L_2(0,l)}.\end{aligned}$$

Now it follows from (5.20), (5.22) that

$$-\varepsilon\langle\mathcal{Q}_\rho\psi_0,\psi_0\rangle-\varepsilon^2\Big\langle G''\mathcal{R}_0(1)G''\frac{\partial\psi_0}{\partial\xi_2},\frac{\partial\psi_0}{\partial\xi_2}\Big\rangle\geq\frac{32\varepsilon^2}{3\pi^2L^3}\|\tilde g\|^2_{L_2(0,l)}.\qquad(5.24)$$

Using (2.1), (5.11), (5.15), (5.16), $\|\frac{\partial\psi_0}{\partial\xi_2}\|=1$, and the inequality

$$\|\mathcal{Q}_\rho\psi_0\|\leq 1+\varepsilon\leq 2,$$

we estimate the remaining terms in the right hand side of (5.19) by noting that

$$\begin{aligned}\varepsilon^3|\langle(G')^2\mathcal{R}_0(\lambda(\rho))\mathcal{Q}_\rho\psi_0,\psi_0\rangle|&\leq\frac{4\varepsilon^3L^2}{\pi^2}\leq\frac{4\varepsilon^3L^4}{\pi^2},\\ \varepsilon^3\Big|\Big\langle G''\mathcal{R}_0(\lambda(\rho))(G')^2\psi_0,\frac{\partial\psi_0}{\partial\xi_2}\Big\rangle\Big|&\leq\frac{2\varepsilon^3L^2}{\pi^2}\leq\frac{2\varepsilon^3L^4}{\pi^2}.\end{aligned}$$

To estimate the penultimate term (5.19) observe that by (5.15), (5.16) and (5.13)

$$\Big|\lambda(\rho)-1)\Big|\leq\varepsilon^2\leq\frac{3\varepsilon}{5000L^5N^2}\leq\frac{3\varepsilon}{5000L^5}$$

since $N\geq 1$. The last estimate, (2.1), and the first inequality in (5.11) yield

$$\varepsilon^2\Big|(\lambda(\rho)-1)\Big\langle G''\mathcal{R}_0(1)\mathcal{R}_0(\lambda(\rho))G''\frac{\partial\psi_0}{\partial\xi_2},\frac{\partial\psi_0}{\partial\xi_2}\Big\rangle\Big|\leq\frac{3\varepsilon^3}{5000L^5}\left(\frac{2L^2}{\pi^2}\right)^2\Big\|\frac{\partial\psi_0}{\partial\xi_2}\Big\|^2_{L_2(\Pi_0)}$$

$$\leq \frac{3\varepsilon^3}{1250\pi^4 L} \leq \frac{3\varepsilon^3 L^4}{1250\pi^4}.$$

We use (5.14) and the estimate

$$\|\mathcal{Q}_\rho \psi_0\| \leq 1 + \varepsilon \leq 2$$

to arrive at

$$\varepsilon^3 \left| \left\langle (\mathrm{I} - \varepsilon \mathcal{Q}_\rho \mathcal{R}_0(\lambda(\rho)))^{-1} \left(\mathcal{Q}_\rho \mathcal{R}_0(\lambda(\rho))\right)^2 \mathcal{Q}_\rho \psi_0, \psi_0 \right\rangle \right| \leq \frac{250 \cdot 104^2 L^4}{125\pi^4 - 8\pi^2} \varepsilon^3.$$

In view of (5.19), (5.24) it leads us to the estimate

$$\lambda(\rho) - 1 \geq \frac{64\varepsilon^2}{(9 + 3\pi^2) L^3} \|\tilde{g}\|^2_{L_2(0,l)} - 225 L^4 \varepsilon^3. \tag{5.25}$$

We note that

$$\frac{64}{9 + 3\pi^2} - 3 \frac{225}{5000} > \frac{3}{2}$$

and use assumption (5.2) to bound (5.25) from below by

$$\frac{64\varepsilon^2}{(9 + 3\pi^2) L^3} \|\tilde{g}\|^2_{L_2(0,l)} - 225 L^4 \varepsilon^2 \frac{3 \|\tilde{g}\|^2_{L_2(0,l)}}{5000 L^7} \geq \frac{3}{2} \|\tilde{g}\|^2_{L_2(0,l)} \frac{\varepsilon^2}{L^3},$$

which completes the proof.

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